Diffeomorphism invariant Colombeau algebras. Part II: Classification

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Abstract

This contribution presents a comprehensive analysis of Colombeau (-type) algebras in the range between the diffeomorphism invariant algebra $\mathcal{G}^d = \mathcal{E}_M^d/\mathcal{N}^d$ introduced in Part I (see [Gro01]) and Colombeau's original algebra $\mathcal{G}^e$ introduced in [Col85]. Along the way, it provides several classification results (again see [Gro01]) which are indispensable for obtaining an intrinsic description of a (full) Colombeau algebra on a manifold ([Gro99]). The latter will be the focus of Part III of this series of contributions.

Key words. Algebras of generalized functions, Colombeau algebras, calculus on infinite dimensional spaces, diffeomorphism invariance.

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1 Introduction

This contribution continues the first in a series of three (Parts I and III also in this volume) by analyzing diffeomorphism invariant Colombeau algebras from a broader point of view. We will use freely notations and results from Part I; for details see [Gro01].

The main result of Section 2 below allows for considerably simplifying the definition of the null ideal: Indeed, it dispenses with taking into account the derivatives of the representative being tested. This applies to virtually all versions of Colombeau algebras. In Section 3 we show that the diffeomorphism invariant algebra $\mathcal{G}^d(\Omega)$ of [Jel99] resp. [Gro01] (see Section 3 of Part I) is not injectively included in the Colombeau algebra $\mathcal{G}^e(\Omega)$ of [Col85] by constructing two counterexamples. Section 4 develops a framework allowing to classify the

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range of algebras which can be positioned between $\mathcal{G}^d(\Omega)$ and (the smooth version of) $\mathcal{G}^e(\Omega)$. In particular, we are going to determine the minimal extent to which the definition of the algebra introduced by J. F. Colombeau and A. Meril in [Col94] has to be modified to obtain diffeomorphism invariance. This leads to the construction of the (diffeomorphism invariant) Colombeau algebra $\mathcal{G}^2(\Omega)$ which is closer to the algebra of [Col94] than the algebra $\mathcal{G}^d(\Omega)$. Certain classification results of Section 4 are essential for obtaining an intrinsic description of Colombeau algebras on manifolds (see Part III resp. [Gro99]).

Both the counterexamples to be constructed in Section 3 will take the form of infinite series, being absolutely convergent in each derivative. Thus we need a theorem guaranteeing the completeness of $\mathcal{E}(\Omega) = \mathcal{C}^\infty(U(\Omega), \mathbb{C}) \equiv \mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega, \mathbb{C})$ with respect to the corresponding topology. To this end, let E, F be locally convex spaces and U an open subset of E. If $f : U \to F$ is smooth, its n-th differential $\mathrm{d}^n f$ belongs to $\mathcal{C}^\infty(U, L^n(E^n, F))$ where $L^n(E^n, F)$ denotes the space $L(E, \ldots, E; F)$ of n-linear bounded maps from $E \times \cdots \times E$ (n factors) into F. (For $n = 0$, set $L^n(E^n, F) := F$.) On $\mathcal{C}^\infty(U, L^n(E^n, F))$, let τ^n_{cb} denote the topology of uniform F-convergence on subsets of the form $K \times B$ where K is a compact subset of U and B is bounded in $E^n = E \times \cdots \times E$. Let $\mathcal{C}^\infty(U, F)$ carry the initial (locally convex) topology τ^∞ induced by the family $(\mathrm{d}^n, \mathcal{C}^\infty(U, L^n(E^n, F)), \tau^n_{cb})_{n \geq 0}$, i.e., the topology of uniform convergence of all derivatives (that is to say, differentials) on sets $K \times B$ as above. Note that on $\mathcal{C}^\infty(\mathbb{R}, F)$, τ^∞ is just the usual Fréchet topology of compact convergence in all derivatives. For the proof of the following theorem, see [Gro01].

1.1 Theorem. *Let E, F be locally convex spaces, assume F to be complete and let U be an open subset of E. Then $\mathcal{C}^\infty(U, F)$ is complete with respect to the topology τ^∞ of uniform F-convergence of all differentials on subsets of the form $K \times B$ where K is a compact subset of U and B is bounded in the appropriate product $E^n = E \times \cdots \times E$. Moreover, for each $p \in \mathbb{N}$, the operator $\mathrm{d}^p : \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, L^p(E^p, F))$ is continuous if both the domain and the range space carry the respective topology τ^∞.*

Now let U denote a (non-empty) open subset of a closed affine subspace E_1 of some locally convex space E, E_0 the linear subspace parallel to E_1 and F a complete locally convex space. *Mutatis mutandis*, 1.1 is valid also in this slightly more general situation. The vectors $v_1, \ldots, v_n$ to be plugged into $\mathrm{d}^n f(x)$ now have to be taken from E_0, as well as B has to denote a bounded subset of E_0^n.

In the following, we will abbreviate $R \circ S^{(\varepsilon)}$ as R_ε, throughout. Terms of the form $\partial^\alpha \mathrm{d}_1^k R_\varepsilon$ always are to be read as $\partial^\alpha \mathrm{d}_1^k(R_\varepsilon)$.

2 A simple condition equivalent to negligibility

The principal part of this section refers to $\mathcal{G}^d$. However, in the concluding remarks we will indicate that the main result is true for virtually all types of Colombeau algebras.

Th. 18 (2°) of [Jel99] gives a condition equivalent to negligibility replacing the term $\partial^\alpha(R(S_\varepsilon\phi(\varepsilon,x),x))$ occurring in the definition (3.1 in Part I resp. Def. 7.3 in [Gro01]) by $(\partial^\alpha \mathrm{d}_1^k R_\varepsilon)(\varphi,x)(\psi_1,\dots,\psi_k)$. (The analogue of this theorem for the case of moderateness can be looked up as 3.2 in Part I.) Moreover, Th. 18 (1°) of [Jel99] shows that we still get a condition equivalent to $R \in \mathcal{N}(\Omega)$ if we simply omit the differential with respect to the first variable φ from (2°), provided R is assumed to be moderate. In the following, we are going to show that a further simplification is possible which might seem rather drastic at first glance: It is not even necessary to consider partial derivatives with respect to $x \in \Omega$. In order to facilitate comparing the conditions mentioned so far we include all of them in the following theorem, though only (0°) is new.

2.1 Theorem. *For $R \in \mathcal{E}_M(\Omega)$, each of the following conditions is equivalent to $R \in \mathcal{N}(\Omega)$:*

(0°) $\forall K \subset\subset \Omega\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B$ *(bounded)* $\subseteq \mathcal{D}(\mathbb{R}^s)$:

$$R_\varepsilon(\varphi,x) = O(\varepsilon^n) \qquad (\varepsilon \to 0).$$

(1°) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B$ *(bounded)* $\subseteq \mathcal{D}(\mathbb{R}^s)$:

$$\partial^\alpha R_\varepsilon(\varphi,x) = O(\varepsilon^n) \qquad (\varepsilon \to 0).$$

(2°) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \forall k \in \mathbb{N}_0\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B$ *(bounded)* $\subseteq \mathcal{D}(\mathbb{R}^s)$:

$$\partial^\alpha \mathrm{d}_1^k R_\varepsilon(\varphi,x)(\psi_1,\dots,\psi_k) = O(\varepsilon^n) \qquad (\varepsilon \to 0).$$

In each of the preceding conditions, the estimate is to be understood as to hold uniformly with respect to $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$ ((1°),(2°)), $\psi_1,\dots,\psi_k \in B \cap \mathcal{A}_{q0}(\mathbb{R}^s)$ ((2°)).

Proof. To highlight the part of the theorem which is new as compared to Th. 18 of [Jel99] we present the proof of (0°) $\Rightarrow$ (1°). To this end, we will show, assuming $R \in \mathcal{E}_M(\Omega)$ to satisfy (0°), that R satisfies (1°) for $\alpha := e_i$, i.e., $\partial^\alpha = \partial_i$ $(i = 1,\dots,s)$ and that, in addition, $\partial_i R$ again is moderate and satisfies (0°). Then it will follow by induction that (1°) holds for all $\alpha \in \mathbb{N}_0^s$.

So suppose $R \in \mathcal{E}_M(\Omega)$ to satisfy (0°) and let $K \subset\subset \Omega$ and $n \in \mathbb{N}$ be given. For $\delta := \min(1, \mathrm{dist}(K,\partial\Omega))$, set $L := K + \overline{B}_{\frac{\delta}{2}}(0)$. Then $K \subset\subset L \subset\subset \Omega$. Now by moderateness of R and Th. 17 of [Jel99] (3.2 of Part I), choose $N \in \mathbb{N}$ such that for every bounded subset B of $\mathcal{D}(\mathbb{R}^s)$ the relation $\partial_i^2 R_\varepsilon(\varphi,x) = O(\varepsilon^{-N})$ as $\varepsilon \to 0$ holds, uniformly for $x \in L$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^s)$. Next, by the assumption of (0°) to hold for R, choose $q \in \mathbb{N}$ such that, again for every bounded subset B of $\mathcal{D}(\mathbb{R}^s)$, we have $R_\varepsilon(\varphi,x) = O(\varepsilon^{2n+N})$ as $\varepsilon \to 0$, uniformly for $x \in L$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$. Now suppose a bounded subset B of $\mathcal{D}(\mathbb{R}^s)$ to be given; let $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$, $x \in K$ and $0 < \varepsilon < \frac{\delta}{2}$; hence $x + \varepsilon^{n+N} e_i \in L$. By Taylor's Theorem, we conclude (to be precise, separately for the real and imaginary part of R)

$$R_\varepsilon(\varphi, x + \varepsilon^{n+N} e_i) \quad = \quad R_\varepsilon(\varphi,x) + \partial_i R_\varepsilon(\varphi,x)\varepsilon^{n+N} + \frac{1}{2}\partial_i^2 R_\varepsilon(\varphi,x_\theta)\varepsilon^{2n+2N}$$

where $x_\theta = x+\theta\varepsilon^{n+N} e_i$ for some $\theta \in (0,1)$; note that also $x_\theta \in L$. Consequently,

$$\partial_i R_\varepsilon(\varphi, x) \quad = \quad \underbrace{\big(R_\varepsilon(\varphi, x+\varepsilon^{n+N} e_i) - R_\varepsilon(\varphi, x)\big)}_{O(\varepsilon^{2n+N})} \varepsilon^{-n-N} - \underbrace{\frac{1}{2}\partial_i^2 R_\varepsilon(\varphi, x_\theta)}_{O(\varepsilon^{-N})} \varepsilon^{n+N},$$

uniformly for $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$, $x \in K$. Having demonstrated $\partial_i R_\varepsilon(\varphi, x) = O(\varepsilon^n)$ for all $i = 1, \ldots, s$, observe that $\partial_i(R_\varepsilon) = (\partial_i R)_\varepsilon$. Therefore, $\partial_i R$ again satisfies (0°). According to Th. 7.10 of [Gro01] (which is non-trivial, see the discussion in Section 7.3 of [Gro01]), $\partial_i R$ is also moderate . By the remark made above, this completes the proof. $\square$

The reader acquainted with E. Landau's paper [Lan14] will easily recognize the method employed therein to form the basis of the preceding proof.

The seemingly technical difference between (0°) and the remaining conditions has decisive effects on applications: For example, if the uniqueness of a solution of a differential equation is to be shown one supposes R_1, R_2 to be representatives of solutions. Note that this includes the assumption that $R_1, R_2 \in \mathcal{E}_M(\Omega)$, hence 2.1 may be applied. For $[R_1] = [R_2]$ in $\mathcal{G}(\Omega)$ we have to show that $R := R_1 - R_2 \in \mathcal{N}(\Omega)$. Now it suffices to check condition (0°) rather than (1°) (resp. (2°) resp. the original definition of $R \in \mathcal{N}(\Omega)$), i.e., there is no need to analyze the behaviour of any derivative of R.

The part of 2.1 saying that for moderate functions (the appropriate analog of) condition (0°) is equivalent to negligibility applies to virtually all versions of Colombeau algebras of practical importance, in particular, to the following:

- For the special algebra as defined, e.g., in [Obe92], p. 109, just replace the term $R_\varepsilon(\varphi, x)$ in condition (0°) by $u_\varepsilon(x)$.
- For the classical full Colombeau algebra of [Col85] simply drop the uniformity requirement concerning φ from (0°).
- For the diffeomorphism invariant Colombeau algebra $\mathcal{G}^2(\Omega)$ to be introduced in Section 4, the corresponding result is stated as Th. 17.9 in [Gro01].
- For the special algebra on smooth manifolds the corresponding result follows from the local characterization of generalized functions (see [Ste00], 4.4).
- The latter also applies to the intrinsically defined full Colombeau algebra on manifolds ([Gro99], Cor. 4.5).

In the first and second of these four instances, the respective proofs are obtained by appropriately slimming down the proof of 2.1.

3 Non-injectivity of the canonical homomorphism from $\mathcal{G}^d(\Omega)$ into $\mathcal{G}^e(\Omega)$

For every open subset Ω of $\mathbb{R}^s$, there is a canonical algebra homomorphism Φ from the diffeomorphism invariant Colombeau algebra $\mathcal{G}^d(\Omega)$ of [Jel99] (see Section 3 of Part I) to the "classical" (full) Colombeau algebra $\mathcal{G}^e(\Omega)$ introduced in [Col85], 1.2.2 (see Section 1 of Part I). By constructing suitable (counter)examples, we are going to show that Φ is not injective in general.

By superscripts d, e we will distinguish between ingredients for constructing $\mathcal{G}^d$ resp. $\mathcal{G}^e$. As in Section 3 of Part I we will use the C-formalism also in the present context. To see that $\mathcal{E}^d$ is a subset of $\mathcal{E}^e$ we have to pass from C-representatives to J-representatives: Smoothness of $R^d \in \mathcal{E}^d$, by definition, is equivalent to smoothness of $(T^*)^{-1}R^d \in \mathcal{C}^\infty(\mathcal{A}_0(\Omega)\times\Omega)$ while for $R^e \in \mathcal{E}^e$, smoothness of $x \mapsto R^e(\varphi, x)$ is equivalent to smoothness of $x \mapsto (T^*)^{-1}R^e(\varphi(.-x), x)$. From this it is clear that $\mathcal{E}^d \subseteq \mathcal{E}^e$. Moreover, we obtain $\mathcal{E}^d_M \subseteq \mathcal{E}^e_M$ and $\mathcal{N}^d \subseteq \mathcal{N}^e$. This follows easily by inspecting the corresponding definitions.

Thus we obtain a canonical map $\Phi : \mathcal{G}^d(\Omega) \to \mathcal{G}^e(\Omega)$ which is an algebra homomorphism respecting the embeddings of $\mathcal{D}'(\Omega)$ and differentiation.

3.1 Remark. (i) Colombeau's original construction in 1.2.2 of [Col85] produces a full algebra $\mathcal{G}^e_1(\Omega)$ differing slightly from $\mathcal{G}^e(\Omega)$ used above. $\mathcal{G}^e_1(\Omega)$ is obtained on the basis of $U_1(\Omega) := T^{-1}(\mathcal{A}_1(\Omega)\times\Omega)$ rather than $U(\Omega) = T^{-1}(\mathcal{A}_0(\Omega)\times\Omega)$. The restriction operator Φ_0 maps $\mathcal{E}^d$ into $\mathcal{E}^e_1$, $\mathcal{E}^d_M$ into $\mathcal{E}^e_{1,M}$ and $\mathcal{N}^d$ into $\mathcal{N}^e_1$, respectively. The canonical map $\Phi_1 : \mathcal{G}^d \to \mathcal{G}^e_1$ induced by Φ_0 acts on representatives as restriction from $T^{-1}(\mathcal{A}_0(\Omega)\times\Omega)$ to $T^{-1}(\mathcal{A}_1(\Omega)\times\Omega)$.
(ii) The counterexamples to be constructed below will settle the question of injectivity not only of $\Phi : \mathcal{G}^d \to \mathcal{G}^e$ but also of $\Phi_1 : \mathcal{G}^d \to \mathcal{G}^e_1$: Φ_1 is injective if and only if Φ is, due to the canonical map $\Psi : \mathcal{G}^e \to \mathcal{G}^e_1$ being injective.

In the following, we will define maps $P, Q : U(\mathbb{R}) \to \mathbb{C}$ each of which satisfies the following conditions (i)–(iv), thereby providing a counterexample to the conjecture of the canonical map Φ being injective.

(i) $R \in \mathcal{E}^d$, i.e., R has to be smooth;

(ii) $R \in \mathcal{E}^d_M$,

(iii) $R \notin \mathcal{N}^d$,

(iv) $R \in \mathcal{N}^e$.

Let $s := 1$, $\Omega := \mathbb{R}$. As a prerequisite we introduce the following notation:

$$\begin{array}{lrcll}
 & \langle\varphi|\varphi\rangle & := & \int \varphi(\xi)\overline{\varphi(\xi)}\,d\xi & (\varphi \in \mathcal{D}(\mathbb{R})) \\
v_k \in \mathcal{D}'(\mathbb{R}): & \langle v_k, \varphi\rangle & := & \int \xi^k \varphi(\xi)\,d\xi & (\varphi \in \mathcal{D}(\mathbb{R}),\ k \in \mathbb{N}_0) \\
v_{\frac{1}{2}} \in \mathcal{D}'(\mathbb{R}): & \langle v_{\frac{1}{2}}, \varphi\rangle & := & \int |\xi|^{\frac{1}{2}} \varphi(\xi)\,d\xi & (\varphi \in \mathcal{D}(\mathbb{R})) \\
 & v(\varphi) & := & \langle\varphi|\varphi\rangle^{\frac{1}{2}} \langle v_{\frac{1}{2}}, \varphi\rangle & (\varphi \in \mathcal{D}(\mathbb{R})) \\
 & g(x) & := & \frac{x}{1+x^2} & (x \in \mathbb{R}) \\
 & e(x) & := & \begin{cases} \exp(-\frac{1}{x}) & (x > 0) \\ 0 & (x \le 0) \end{cases} & (x \in \mathbb{R}) \\
 & \gamma_k & := & k + \frac{1}{k} & (k \in \mathbb{N}).
\end{array}$$

Finally, choose an (even) function $\sigma \in \mathcal{D}(\mathbb{R})$ satisfying $0 \le \sigma \le 1$, $\sigma(x) \equiv 1$ for $|x| \le \frac{1}{2}$, $\sigma(x) \equiv 0$ for $|x| \ge \frac{3}{2}$ and set

$$h_k(x) := \sigma(x) \cdot 2g(x) + (1 - \sigma(x)) \cdot \operatorname{sgn}(x) \cdot |2g(x)|^{\gamma_k} \qquad (x \in \mathbb{R},\ k \in \mathbb{N}).$$

Apart from abbreviating $R \circ S^{(\varepsilon)} = R \circ (S_\varepsilon \times \mathrm{id})$ as R_ε for any function R defined on $\mathcal{A}_0(\mathbb{R}) \times \mathbb{R}$, we also will write R_ε for $R \circ S_\varepsilon$ if R is defined on $\mathcal{A}_0(\mathbb{R})$.

3.2 Definition. *Let $\varphi \in \mathcal{A}_0(\mathbb{R})$, $x \in \mathbb{R}$ and set*

$$\begin{aligned}
P(\varphi, x) &:= \sum_{k=1}^{\infty} \frac{1}{k!} \cdot g\big(\langle\varphi|\varphi\rangle^{\gamma_k} e(v(\varphi))\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle, \\
Q(\varphi, x) &:= \sum_{k=1}^{\infty} \frac{1}{k!} \cdot h_k\big(\langle\varphi|\varphi\rangle^{\frac{3}{2}} \langle v_{\frac{1}{2}}, \varphi\rangle\big) \cdot \langle\varphi|\varphi\rangle^{\gamma_k} \cdot \langle v_k, \varphi\rangle.
\end{aligned}$$

Hence P and Q, in fact, only depend on φ. Explicitly, P is given by

$$P(\varphi, x) = \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \frac{\left(\int \varphi(\xi)\overline{\varphi(\xi)}\,d\xi\right)^{k+\frac{1}{k}} \exp\left(-\frac{1}{\left(\int \varphi(\xi)\overline{\varphi(\xi)}\,d\xi\right)^{\frac{1}{2}} \left(\int |\xi|^{\frac{1}{2}} \varphi(\xi)\,d\xi\right)}\right)}{1 + \left(\left(\int \varphi(\xi)\overline{\varphi(\xi)}\,d\xi\right)^{k+\frac{1}{k}} \exp\left(-\frac{1}{\left(\int \varphi(\xi)\overline{\varphi(\xi)}\,d\xi\right)^{\frac{1}{2}} \left(\int |\xi|^{\frac{1}{2}} \varphi(\xi)\,d\xi\right)}\right)\right)^2}.$$

It can be shown that the series for both P and Q converge uniformly on bounded subsets of $\mathcal{A}_0(\mathbb{R})$, rendering P and Q well-defined by 1.1. For the proof of claims (i)–(iv) above we refer to [Gro01].

The reader might ask if it is indeed necessary to come up with counterexamples as complicated as P and Q certainly are. The author doubts that easier ones might be possible. This view is based on reflecting on the rôles each of the three factors constituting a single term of the series (for P, say) in fact has to play:

- $\langle v_k, \varphi \rangle$ distinguishes between the spaces $\mathcal{A}_q(\mathbb{R})$; this is crucial for the negligibility properties.

- $\langle \varphi | \varphi \rangle^{\gamma_k} = \langle \varphi | \varphi \rangle^k \cdot \langle \varphi | \varphi \rangle^{\frac{1}{k}}$, on the one hand, after scaling of φ compensates for the factor ε^k generated by scaling φ in $\langle v_k, \varphi \rangle$. On the other hand, it introduces a factor $\varepsilon^{-\frac{1}{k}}$ making the first non-vanishing term of the series the dominant one as $\varepsilon \to 0$.

- $g(\langle \varphi | \varphi \rangle^{\gamma_k} e(\langle v, \varphi \rangle))$ allows the pointwise vs. uniformly distinction being necessary to obtain $P \notin \mathcal{N}^d$, $P \in \mathcal{N}^e$. Though $g(\langle \varphi | \varphi \rangle^{\gamma_k} \langle v, \varphi \rangle)$ would suffice to achieve the latter, this alternative choice for the argument of g would produce, via the chain rule, a factor $\varepsilon^{-n(k+\frac{1}{k})}$ in the k-th term of $\mathrm{d}^n P_\varepsilon$ which would be disastrous for the moderateness of P. The function e (together with $\varepsilon^{-\gamma_k}$ in the argument of g) suppressing this unwanted factor, P becomes moderate in the end.

Similar arguments apply to Q.

4 Classification of smooth Colombeau algebras between $\mathcal{G}^d(\Omega)$ and $\mathcal{G}^e(\Omega)$

Apart from $\mathcal{G}^e(\Omega)$, all algebras to be considered in this section have $\mathcal{C}^\infty(U(\Omega))$ resp. $\mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega)$ as their basic space. In particular, they are smooth algebras in the sense that representatives R have to be smooth also with respect to φ. The term "test object" will always refer to some element of $\mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$.

4.1 Definition. *Let $q \in \mathbb{N}$. A function $\phi : I \to \mathcal{D}(\mathbb{R}^s)$ (possibly depending also on other arguments, e.g., on $x \in \Omega$) is said to have* vanishing moments of order q *if $\int \xi^\alpha \phi(\varepsilon)(\xi)\, d\xi = 0$ for all $\alpha \in \mathbb{N}_0^s$ with $1 \le |\alpha| \le q$. It is said to have* asymptotically vanishing moments of order q *if $\int \xi^\alpha \phi(\varepsilon)(\xi)\, d\xi = O(\varepsilon^q)$ for all $\alpha \in \mathbb{N}_0^s$ with $1 \le |\alpha| \le q$. To which extent this estimate is assumed to hold uniformly with respect to, e.g., $x \in \Omega$ has to be specified separately (see below).*

To obtain a classification of Colombeau algebras lying in the range between $\mathcal{G}^d(\Omega)$ and (the smooth version of) $\mathcal{G}^e(\Omega)$ we introduce symbols of the forms [p], [M], [p, M] where p refers to the parameters and M to the moment properties of a test object. p, being one of c, ε, εx denotes test objects of the form φ ("constant"), $\phi(\varepsilon)$ and $\phi(\varepsilon, x)$, respectively. M, on the other hand, can take the values 0,A,V, corresponding to $\mathcal{A}_0(\mathbb{R}^n)$, asymptotically vanishing moments and $\mathcal{A}_q(\mathbb{R}^n)$, respectively. [A] only applies to parametrization type [ε]. For test objects of type [εx], we distinguish the following uniformity requirements concerning asymptotically vanishing moments:
[$\mathrm{A_l}$]: uniformly on the particular $K \subset\subset \Omega$ ("locally");
[$\mathrm{A_g}$]: uniformly on each $L \subset\subset \Omega$ ("globally");
[$\mathrm{A_l^\infty}$]: all derivatives $\partial_x^\alpha \phi(\varepsilon, x)$ uniformly on the particular $K \subset\subset \Omega$;

$[\mathrm{A}_\mathrm{g}^\infty]$: all derivatives $\partial_x^\alpha \phi(\varepsilon, x)$ uniformly on each $L \subset\subset \Omega$.

Here, "on the particular $K \subset\subset \Omega$" is to be read as "on the particular $K \subset\subset \Omega$ on which R is being tested". If this compact set K and/or the order q of the (asymptotic) vanishing of moments is to be specified, K resp. q will be put as subscript(s) to the corresponding A-symbol, e.g., $[\mathrm{A}_\mathrm{l}]_{K,q}$. If in $[\mathrm{p}, \mathrm{M}]$ M is one of the A-symbols then $\mathrm{p} = \varepsilon$ resp. $\mathrm{p} = \varepsilon x$, being redundant, will be omitted frequently.

If $[X]$ and $[Y]$ are chosen from the set of the eleven types such that $\mathcal{E}_M[X] \subseteq \mathcal{E}_M[Y]$) and if, in addition, $[Y]$ is one of the types [A] or [V] then it easily checked that $\mathcal{E}_M[X]$ is an algebra containing $\mathcal{N}[Y] \cap \mathcal{E}_M[X]$ as an ideal. Consequently, $\mathcal{E}_M[X]/(\mathcal{N}[Y] \cap \mathcal{E}_M[X])$ is an algebra. We shall refer to algebras arising in this way by the term "Colombeau-type algebras". Altogether there are 46 admissible choices of pairs $[X], [Y]$. In the following definition, we will specify eleven algebras of this kind, one for each type of moderateness. These will be the only ones we are to deal with in the sequel. Each of the remaining Colombeau-type algebras can be obtained as some subalgebra or some quotient algebra of one of them. Note, however, that the collection of these eleven algebras is not minimal in this respect (see Th. 17.10 in [Gro01]).

4.2 Definition. *If* $[X]$ *is one of the types* [V] *or* [A] *define*

$$\mathcal{G}[X] := \mathcal{E}_M[X]/\mathcal{N}[X];$$

for types [0] *define*

$$\begin{aligned}
\mathcal{G}[\varepsilon x, 0] &:= \mathcal{E}_M[\varepsilon x, 0]/\big(\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}^\infty] \cap \mathcal{E}_M[\varepsilon x, 0]\big),\\
\mathcal{G}[\varepsilon, 0] &:= \mathcal{E}_M[\varepsilon, 0]/\big(\mathcal{N}[\varepsilon, \mathrm{A}] \cap \mathcal{E}_M[\varepsilon, 0]\big),\\
\mathcal{G}[\mathrm{c}, 0] &:= \mathcal{E}_M[\mathrm{c}, 0]/\big(\mathcal{N}[\mathrm{c}, \mathrm{V}] \cap \mathcal{E}_M[\mathrm{c}, 0]\big).
\end{aligned}$$

We will refer to $\mathcal{G}[X]$ also by "the algebra of type $[X]$". The open set Ω is omitted from the notation. Denoting by $\mathcal{G}_0^e(\Omega)$ the "smooth part" of $\mathcal{G}^e(\Omega)$, i.e., the subalgebra formed by all members having a smooth representative $R \in \mathcal{C}^\infty(U^e(\Omega))$, it is easy to see that $\mathcal{G}_0^e(\Omega) = \mathcal{G}[\mathrm{c}, \mathrm{V}]$. $\mathcal{G}^1(\Omega)$ obviously is equal to $\mathcal{G}[\varepsilon, \mathrm{A}]$; the algebra $\mathcal{G}^2(\Omega)$ to be discussed below is obtained as $\mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{g}^\infty]$. $\mathcal{G}^d(\Omega)$, finally, is given as $\mathcal{G}[\varepsilon x, 0]$. Observe that according to Th. 7.9 of [Gro01] (3.4 in Part I), $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}^\infty]$ can be replaced by $\mathcal{N}[\varepsilon x, \mathrm{V}]$ in the definition of $\mathcal{G}[\varepsilon x, 0]$.

Cor. 16.8 of [Gro01] shows that test objects of types $[\mathrm{A}_\mathrm{g}]$ and $[\mathrm{A}_\mathrm{g}^\infty]$, respectively, give rise to the same moderate resp. negligible functions. Moreover, by Cor. 17.6 of [Gro01] also test objects of type $[\mathrm{A}_\mathrm{l}^\infty]$ lead to the same respective notions of moderateness and negligibility as test objects of type $[\mathrm{A}_\mathrm{g}^\infty]$ do. This actually leaves us with nine possibly different algebras.

As to the diagram formed by the canonical homomorphisms between these nine algebras, note that there is no such mapping from $\mathcal{G}^d(\Omega) = \mathcal{G}[\varepsilon x, 0]$ into $\mathcal{G}[\varepsilon x, \mathrm{A}_\mathrm{l}]$ since $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}] \cap \mathcal{E}_M[\varepsilon x, 0]$—not containing any of the functions $R(\varphi, x) := \int \xi^\beta \varphi(\xi)\, d\xi$—is strictly smaller than $\mathcal{N}[\varepsilon x, \mathrm{A}_\mathrm{l}^\infty] \cap \mathcal{E}_M[\varepsilon x, 0]$. We

do have canonical homomorphisms, however, both from $\mathcal{G}^d(\Omega) = \mathcal{G}[\varepsilon x, 0]$ and from $\mathcal{G}[\varepsilon x, \mathrm{A_l}]$ into $\mathcal{G}^2(\Omega) = \mathcal{G}[\varepsilon x, \mathrm{A_g^\infty}]$. So we finally arrive at

$$
\begin{array}{ccccccc}
 & & \mathcal{G}[\varepsilon x, 0] & \rightarrow & \mathcal{G}[\varepsilon, 0] & \rightarrow & \mathcal{G}[\mathrm{c}, 0] \\
 & & \downarrow & & \downarrow & & \\
\mathcal{G}[\varepsilon x, \mathrm{A_l}] & \rightarrow & \mathcal{G}[\varepsilon x, \mathrm{A_g^\infty}] & \rightarrow & \mathcal{G}[\varepsilon, \mathrm{A}] & & \downarrow \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{G}[\varepsilon x, \mathrm{V}] & \rightarrow & \mathcal{G}[\varepsilon, \mathrm{V}] & \rightarrow & \mathcal{G}[\mathrm{c}, \mathrm{V}]
\end{array}
$$

By the methods employed in [Gro01] one can show that each of the nine algebras occurring in the diagram (injectively) contains $\mathcal{D}'(\Omega)$ via ι; with one exception (namely, $\mathcal{G}[\varepsilon x, \mathrm{A_l}]$; cf. Ex. 7.7 in [Gro01]) the restriction of $\iota : \mathcal{D}' \to \mathcal{G}$ to $\mathcal{C}^\infty$ coincides with $\sigma : \mathcal{C}^\infty \to \mathcal{G}$, implying that ι preserves the product of smooth functions, see [Gro01] for details and proofs. Moreover, for each type $[X]$ except $[\varepsilon x, \mathrm{A_l}]$, $\mathcal{E}_M$ and $\mathcal{N}$ are invariant under differentiation, thus rendering $\mathcal{G}[X]$ a differential algebra. Concerning diffeomorphism invariance, finally, one can show that $\mathcal{G}^d = \mathcal{G}[\varepsilon x, 0]$, $\mathcal{G}^2 := \mathcal{G}[\varepsilon x, \mathrm{A_g^\infty}]$ and $\mathcal{G}[\varepsilon x, \mathrm{A_l}]$ in fact share this property, yet neither of the remaining six algebras does. For the proofs of these statements we refer to Chapter 17 of [Gro01]. $\mathcal{G}^2 := \mathcal{G}[\varepsilon x, \mathrm{A_g^\infty}]$ turns out to be the most delicate case in the technical respect.

Summarizing, we obtain that $\mathcal{G}^d(\Omega)$ and $\mathcal{G}^2(\Omega)$ are the only diffeomorphism invariant Colombeau algebras among the eleven (resp. nine) algebras defined in 4.2.

The algebra $\mathcal{G}^2(\Omega)$ of type $[\varepsilon x, \mathrm{A_g^\infty}]$ can be viewed as resulting from the algebra $\mathcal{G}^1(\Omega) = \mathcal{G}[\varepsilon, \mathrm{A}]$ of [Col94] by applying the minimal modification necessary to obtain diffeomorphism invariance.

The fact that all three types $[\mathrm{A_g^\infty}]$, $[\mathrm{A_g}]$ and $[\mathrm{A_l^\infty}]$ give rise to the same notions of moderateness resp. negligibility, hence to the same Colombeau algebra, constitutes one of the key ingredients for obtaining an intrinsic description of the algebra $\mathcal{G}^d$ on manifolds: The property of a test object living on the manifold to have asymptotically vanishing moments can be formulated in intrinsic terms, indeed (see [Gro99], Def. 3.5 resp. Part III); yet it would be virtually unmanageable to deal with the latter property also for derivatives of this test object, which, of course, are to be understood in this general case as appropriate Lie derivatives with respect to smooth vector fields. Now Cors. 16.8 and 17.6 of [Gro01] allow to dispense with derivatives of test objects as regards the asymptotic vanishing of the moments, provided all $K \subset\subset \Omega$ are taken into account ([Gro99], Cor. 4.5).

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